Gauge theory for discotic liquid crystals

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We present a gauge theory, Abelian but not equivalent to electrodynamics, for a conjectured hexatic $N+6$ phase near the transition to the hexagonal discotic phase. The bond-angle field $\mathbf{\Omega}$, determining the local orientational order for discotic phases, plays the role of a gauge field. We perform an appropriate gauge transformation, by which the bend mode of the director distortion is decoupled from the density-wave order parameter. Furthermore, as a consequence of the performed gauge transformation, the bend term in the free energy becomes nonanalytic in the wave vector. As an exact result, the bend elastic constant K_3, being the coefficient of such a nonanalytic term in the free energy, has no critical enhancement to all orders in perturbation theory.

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I. INTRODUCTION

The melting of the hexagonal discotic phase into the nematic phase was described by a model [1,2] which assumes the existence of an intermediate hexatic $N+6$ phase [1,3]. The hexatic phase is predicted on symmetry grounds, but it is still experimentally undiscovered. Such a phase can be viewed in the framework of bond orientationally ordered phases [3–5] that have been theoretically predicted in a wide class of systems. In fact, the hexatic phase is intermediate between the fully disordered phase and the ordered one, since it is translationally invariant like the nematic phase, but it shows long-range sixfold orientational order around the director like the hexagonal discotic phase. We point out that the translationally ordered phase can be considered as a quasi-two-dimensional system [1], since the sites of the two-dimensional hexagonal lattice are occupied by columnar stacks of disklike molecules, parallel to each other, with liquidlike behavior along the third dimension.

According to the aforementioned model [1], the condensation of the hexagonal two-dimensional lattice from the intermediate $N+6$ phase is described by the onset of a triple mass-density wave in the plane orthogonal to the nematic director (conventionally taken as the $\hat{\mathbf{z}}$ axis). The set of the three complex amplitudes of such a density wave is assumed as the translational order parameter describing the phase transition. In the hexatic phase, the two-dimensional lattice is melted and then the translational order parameter vanishes. The orientational order, on the contrary, is maintained through the transition and then the $N+6$ phase, as well as the ordered phase, exhibits hexagonal anisotropy. Owing to such a sixfold anisotropy, the nematic director $\mathbf{m}$ is not sufficient, in discotic liquid crystals, to fix orientational order. We must also introduce a bond-angle field Ω_z, which gives the local orientation of the two-dimensional lattice in the XY plane. Such Ω_z can be defined as the rotation angle around the unperturbed nematic director $\mathbf{m}_0$ between a given reciprocal lattice vector and a fixed $\hat{\mathbf{x}}$ axis. Actually, the local orientational order of the hexatic phase, as well as that of the hexagonal discotic phase, is determined by the full rotation field [1]:

$$\mathbf{\Omega}=\Omega_z\mathbf{m}_0+(\mathbf{m}_0\times\delta\mathbf{m}) , \tag{1.1}$$

where $\delta\mathbf{m}$ is a small distortion of $\mathbf{m}_0$, and then it is orthogonal to $\mathbf{m}_0\equiv\hat{\mathbf{z}}$. In the uniform configuration, the field $\mathbf{\Omega}$ is constant and then can be taken vanishing (i.e., $\Omega_z=0,\delta\mathbf{m}=\mathbf{0}$).

As for smectic liquid crystals [6], free energy must be invariant under global uniform rotations of the system, in this case under simultaneous rotations of the liquid columns and of the director, as well as under simultaneous torsions of the columns and Ω_z rotations of the two-dimensional lattice. The requirement of invariance under global uniform rotations modifies the gradient terms in the free energy, so that it becomes a local rotation invariance [1,4,6]. Such an invariance can be seen as a local gauge invariance, i.e., invariance under nonuniform local rotations, anyway different, in form, from smectic gauge invariance that is fully analogous with electrodynamic gauge invariance. The constraint of local invariance under rotations [1,4] yields the coupling between the translational order parameter and the local field $\mathbf{\Omega}$, which can be therefore considered as a gauge field. The gradient terms are thus analogous to covariant derivatives, and they contain the gauge coupling between the gauge field $\mathbf{\Omega}$ and the matter, represented by the density-wave order parameters. The full elastic energy associated with the strains of $\mathbf{\Omega}$ was calculated in [2], and it corresponds to the pure gauge field energy in electrodynamics or in other gauge theories.

However, local rotation invariance of discotics is not exact, being broken by the bend term. In fact, a configuration where the liquid columns are bent, which is the result of a local nonuniform rotation, requires bend energy with respect to the flat configuration, if the director remains locally parallel to the columns. Therefore, only invariance under uniform rotations remains exact. The bend mode in discotics then plays the same role that

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the splay mode plays in smectic liquid crystals. In particular, the hexagonal discotic phase is rigid against twist and splay deformations, because only bend deformations are compatible with an undistorted two-dimensional lattice [7].

As a consequence of the coupling between the order parameter and $\mathbf{\Omega}$, the critical fluctuations of the order parameter drive the critical enhancements of the Frank elastic constants, in the hexatic phase near the transition to the hexagonal phase [1]. In smectic liquid crystals, a similar coupling between the order parameter and the director yields the critical enhancements of the Frank elastic constants [8]. Actually, in discotic liquid crystals, only splay and twist elastic constants have critical divergent contributions, while the bend mode is noncritical [1]. In fact, the bend mode of the director distortions should not be coupled to the mass-density wave order parameter, as suggested by the above-mentioned distinctive feature of the bend deformations that can spread through the system without affecting the local structure of the two-dimensional lattice [7]. In a similar way, the splay mode is noncritical for smectic phases [8].

The aim of this paper is just to show that it is possible, by means of an appropriate gauge transformation, to separate the director gauge field $\delta\mathbf{m}$ into the superposition of a soft-bend mode and a critical mode. The soft-bend mode, by the suitable gauge transformation, can be decoupled from the density-wave order parameter. As a consequence, the bend elastic constant K_3 has no critical enhancement. In fact, the coupling between the order parameter and the gauge field $\delta\mathbf{m}$ renormalizes the Frank elastic constants. The divergence of the order-parameter correlation length near the phase transition, in the hexatic phase, drives the divergence of the splay and twist elastic constants, K_1 and K_2, respectively. The bend component, being decoupled from the matter, is therefore noncritical.

We stress the point that the bend mode breaks local gauge invariance, for discotic phases. As a consequence, bend can be decoupled from the translational order parameter by a gauge transformation, so implying noncritical behavior of bend. In smectics liquid crystals, a similar gauge transformation, which gives a gauge corresponding to the transverse gauge in electrodynamics, decouples the splay noncritical mode from the matter [9]. Anyway, the particular gauge coupling of our model [1] yields an Abelian gauge theory not equivalent in form to electrodynamics, and therefore it is intriguing to investigate its formal properties. A somewhat similar gauge coupling has been proposed for the conjectured cubic-liquid-crystal phase [4].

As a consequence of the above-described gauge transformation, the propagator of the gauge field $\delta\mathbf{m}$, in terms of what we will call the soft-bend gauge, exhibits nonanalytic dependence on the wave vector. The coefficient of such a nonanalytic term in the free energy is just the bend elastic constant K_3. Perturbation theory in the vertex gauge interaction yields a series of graphs that renormalize the Frank elastic constants. Therefore, the bend constant K_3 is not critically enhanced to all orders in perturbation theory, since it is the coefficient of a nonanalytic term in the free energy which cannot be reproduced by the perturbation expansion. For an analogous reason, the splay constant K_1 is not renormalized in smectic phases [9,10]. We notice that the proposed gauge transformation, on the one hand, decouples bend from the matter and, on the other hand, introduces nonanalytic properties in the bend term of the free energy, so giving self-consistent results.

Actually, the decoupling gauge transformation shows a slightly different feature in Fourier space, with respect to usual coordinate space. In coordinate space, it is possible to separate the director field $\delta\mathbf{m}$ into the superposition of a pure bend mode and a critical mode that is composed of only twist and splay modes. On the contrary, in Fourier space, besides the pure soft-bend mode, we obtain a critical mode that is a mixing of splay, twist, and bend component too (see also [11]). That is due to our peculiar gauge coupling, and it has no analog in smectic gauge invariance. Nevertheless, such a residual bend component in the critical field is not effective to enhance critically the bend constant.

This paper is divided into three sections of which this is the first. In Sec. II, we define the general gauge transformation under which the free energy of our model [1,2] is (indeed, not strictly) invariant. Then we present the particular gauge transformation by which the bend mode is decoupled from the order parameter; finally, we write the free energy in the new gauge, which we call the soft-bend gauge, both in coordinate space and in Fourier space. In Sec. III, the propagator of the gauge field, in the soft-bend gauge, is drawn, and it is shown that, because of a term nonanalytic in the wave vector, the bend constant is noncritical.

II. GAUGE TRANSFORMATIONS IN DISCOTIC PHASES

The order parameter for the condensation of the two-dimensional hexagonal lattice from the hexatic $N+6$ phase is a triple mass-density wave [1]:

$$\delta\rho(\mathbf{r})=\sum_{i=1}^{3}\eta_i(\mathbf{r})\exp(i\mathbf{q}_i\cdot\mathbf{r})+\text{c.c.} , \tag{2.1}$$

where η_i ($i=1, 2, 3$) are the complex local amplitudes and $\mathbf{q}_i$ are the shortest reciprocal lattice vectors. The full free energy of the model, in terms of the local rotation field $\mathbf{\Omega}$ defined in Eq. (1.1) and of the local order parameter η_i in Eq. (2.1), is [1,2]

$$F=F_0+F_{\text{el}} \tag{2.2a}$$

with

$$F_0=\int d^3r\left\{a\sum_{i=1}^{3}|\eta_i(\mathbf{r})|^2+b\eta_1(\mathbf{r})\eta_2(\mathbf{r})\eta_3(\mathbf{r})+c_1\left[\sum_{i=1}^{3}|\eta_i(\mathbf{r})|^2\right]^2+c_2\sum_{i=1}^{3}|\eta_i(\mathbf{r})|^4\right.$$
$$\left.+\frac{1}{2M_\parallel}\sum_{i=1}^{3}\left|[\nabla_z+i\mathbf{q}_i\cdot\delta\mathbf{m}(\mathbf{r})]\eta_i(\mathbf{r})\right|^2+\frac{1}{2M_\perp}\sum_{i=1}^{3}\left|[\nabla_\perp-i\Omega_z(\mathbf{r})\mathbf{m}_0\times\mathbf{q}_i]\eta_i(\mathbf{r})\right|^2\right\} \tag{2.2b}$$

and

$$F_{\rm el}=\tfrac{1}{2}\int d^3r\{K_1(\mathrm{div}\delta\mathbf{m})^2+K_2(\mathbf{m}_0\cdot\mathrm{rot}\delta\mathbf{m})^2+K_3(\mathbf{m}_0\times\mathrm{rot}\delta\mathbf{m})^2+\gamma_1(\nabla_z\Omega_z)^2+\gamma_2(\nabla_\perp\Omega_z)^2+2\gamma_3(\mathbf{m}_0\cdot\mathrm{rot}\delta\mathbf{m})(\nabla_z\Omega_z)\}\ , \tag{2.2c}$$

where, as usual in Landau free energy, $a=a_0(T-T^*)$, T^* being a second-order phase-transition temperature, while the other coefficients are positive constants. The free energy (2.2) is written in the "natural" gauge, which we call the discotic gauge, and it explicitly shows the coupling between the gauge field $\mathbf{\Omega}$ and the order parameter η_i, in the order-parameter gradient terms.

For our model free-energy, one can define a general gauge transformation:

$$\eta_i(\mathbf{r})=e^{-i\phi_i(\mathbf{r})}\eta_i'(\mathbf{r})\ , \tag{2.3a}$$

$$\delta\mathbf{m}(\mathbf{r})=\mathbf{A}(\mathbf{r})+\mathbf{B}(\mathbf{r})\ , \tag{2.3b}$$

$$\Omega_z(\mathbf{r})=\Omega_z'(\mathbf{r})+\Lambda(\mathbf{r})\ , \tag{2.3c}$$

with

$$\sum_{i=1}^{3}\phi_i(\mathbf{r})=0\ , \tag{2.4a}$$

$$\nabla_z\phi_i(\mathbf{r})=\mathbf{q}_i\cdot\mathbf{B}(\mathbf{r})\ , \tag{2.4b}$$

and

$$\nabla_\perp\phi_i(\mathbf{r})=-(\mathbf{m}_0\times\mathbf{q}_i)\Lambda(\mathbf{r})\ . \tag{2.4c}$$

Note that Eq. (2.4a) is compatible with Eqs. (2.4b) and (2.4c) because of $\sum_{i=1}^{3}\mathbf{q}_i=\mathbf{0}$, for the hexagonal two-dimensional lattice. Inserting Eqs. (2.3) in the free energy (2.2), we get the free energy in the new gauge, expressed in terms of the transformed local fields $\eta_i'(\mathbf{r})$, $\mathbf{A}(\mathbf{r})$, and $\Omega_z'(\mathbf{r})$. Actually, one can verify that, owing to Eqs. (2.4), the self-energy and the self-interaction of the order-parameter fields η_i, as well as the gradient terms of η_i, are invariant under the above defined gauge transformation. In fact, the free energy (2.2b) takes the same form, both in the discotic gauge variables and in the transformed field variables. On the contrary, the Frank energy, Eq. (2.2c), is not strictly gauge invariant, being dependent, in general, on the gauge-transformation functions $\mathbf{B}(\mathbf{r})$ and $\Lambda(\mathbf{r})$.

In order to decouple the bend field from the order parameter η_i, we restrict the general gauge transformation defined in Eqs. (2.3) and (2.4) to the particular case

$$\phi_i=\phi_i(z)\ , \tag{2.5a}$$

$$\mathbf{m}_0\times\mathrm{rot}\,\mathbf{A}=\mathbf{0}\ . \tag{2.5b}$$

If the order-parameter phase-shift depends only on the coordinate z, as in Eq. (2.5a), then the gauge-transformation function $\mathbf{B}$ is a pure bend field. In fact, inverting Eq. (2.4b) by means of

$$\sum_{i=1}^{3}q_{i\alpha}q_{i\beta}=\tfrac{3}{2}q_0^2\delta_{\alpha\beta}\ ,\quad \alpha,\beta=x,y\ , \tag{2.6}$$

i.e., Eq. (6) of [1], we get

$$\mathbf{B}(z)=\tfrac{2}{3}q_0^{-2}\sum_{i=1}^{3}\frac{\partial\phi_i(z)}{\partial z}\mathbf{q}_i\ , \tag{2.7}$$

where q_0 is the modulus of $\mathbf{q}_i$, proportional to the inverse lattice spacing. As the $\mathbf{q}_i$ vectors lie in the XY plane, and the $\mathbf{B}$ vector in (2.7) depends only on z, we have

$$\mathrm{div}\mathbf{B}=0 \tag{2.8}$$

and

$$\mathbf{m}_0\cdot\mathrm{rot}\mathbf{B}=0\ . \tag{2.9}$$

Equations (2.8) and (2.9) mean that the splay and the twist components of the field $\mathbf{B}$ vanish, respectively. On the other hand, Eq. (2.5b) means that the bend component of $\mathbf{A}$ vanishes, so that

$$\mathrm{div}\delta\mathbf{m}=\mathrm{div}\,\mathbf{A}\ , \tag{2.10a}$$

$$\mathbf{m}_0\cdot\mathrm{rot}\delta\mathbf{m}=\mathbf{m}_0\cdot\mathrm{rot}\,\mathbf{A}\ , \tag{2.10b}$$

$$\mathbf{m}_0\times\mathrm{rot}\delta\mathbf{m}=\mathbf{m}_0\times\mathrm{rot}\mathbf{B}\ . \tag{2.10c}$$

In that way, the director field $\delta\mathbf{m}$ has been separated, as in Eq. (2.3b), into the superposition of a pure bend mode $\mathbf{B}$ and a gauge field $\mathbf{A}$, which contains only splay and twist components. Moreover, as the phase shift ϕ depends only on z, from Eq. (2.4c) we get $\Lambda=0$ and then Eq. (2.3c) gives $\Omega_z'=\Omega_z$, i.e., the proposed transformation only changes the gauge field $\delta\mathbf{m}$ into $\mathbf{A}$, while not affecting Ω_z.

The free energy in the soft-bend gauge is thus

$$F=\int d^3r\left\{a\sum_{i=1}^{3}|\eta_i'|^2+b\eta_1'\eta_2'\eta_3'+c_1\left[\sum_{i=1}^{3}|\eta_i'|^2\right]^2+c_2\sum_{i=1}^{3}|\eta_i'|^4+\frac{1}{2M_\parallel}\sum_{i=1}^{3}\left|[\nabla_z+i\mathbf{q}_i\cdot\mathbf{A}]\eta_i'\right|^2\right.$$
$$+\frac{1}{2M_\perp}\sum_{i=1}^{3}\left|[\nabla_\perp-i\Omega_z\mathbf{m}_0\times\mathbf{q}_i]\eta_i'\right|^2+\tfrac{1}{2}K_1(\operatorname{div}\mathbf{A})^2+\tfrac{1}{2}K_2(\operatorname{rot}\mathbf{A})^2+\tfrac{1}{2}\gamma_1(\nabla_z\Omega_z)^2$$
$$\left.+\tfrac{1}{2}\gamma_2(\nabla_\perp\Omega_z)^2+\gamma_3(\operatorname{rot}\mathbf{A})(\nabla_z\Omega_z)\right\}+\tfrac{1}{2}K_3\int d^3r(\operatorname{rot}\mathbf{B})^2 \ . \tag{2.11}$$

Therefore, in the soft-bend gauge, the order-parameter gradient terms contain only the field $\mathbf{A}$, which is a superposition of only splay and twist modes, while the pure bend field $\mathbf{B}$ is decoupled from the order parameter η_i. As we shall see in Sec. III, the decoupling of the bend mode from the order parameter implies noncritical behavior of the bend constant. In this sense, bend is a soft mode, in discotic liquid crystals. Equation (2.11), by comparison with Eq. (2.2), also shows that the bend mode explicitly breaks local gauge invariance. In fact, the free energy in the soft-bend gauge, Eq. (2.11), takes the same form as in the discotic gauge, Eq. (2.2), with the transformed fields η_i' and $\mathbf{A}$ in place of the fields η_i and $\delta\mathbf{m}$, except for the last term in Eq. (2.11), which is a bend term, depending on the performed gauge transformation. Therefore, the fact that bend breaks local gauge invariance is strictly related to decoupling of bend from the matter.

In Fourier space, Frank elastic energy (2.2c) can be written as (see also [12])

$$F_{\rm el}=\tfrac{1}{2}\int\frac{d^3q}{(2\pi)^3}\{(K_1q_\perp^2+K_3q_z^2)|\delta m_\perp(\mathbf{q})|^2+(K_2q_\perp^2+K_3q_z^2)|\delta m_t(\mathbf{q})|^2+(\gamma_1q_z^2+\gamma_2q_\perp^2)|\Omega_z(\mathbf{q})|^2$$
$$+\gamma_3q_zq_\perp[\Omega_z(\mathbf{q})\delta m_t^*(\mathbf{q})+\text{c.c.}]\} \ , \tag{2.12}$$

where $\mathbf{q}_\perp$ is the projection of the wave vector $\mathbf{q}$ on the XY plane, while $\delta m_\perp(\mathbf{q})$ and $\delta m_t(\mathbf{q})$ are the components of $\delta\mathbf{m}(\mathbf{q})$, longitudinal and transverse to $\mathbf{q}_\perp$, respectively.

Performing a general gauge transformation, in Fourier space, we have

$$\delta\mathbf{m}(\mathbf{q})=\mathbf{A}(\mathbf{q})+\mathbf{B}(\mathbf{q}) \ , \tag{2.13}$$

with

$$\mathbf{A}(\mathbf{q})=A_q(\mathbf{q})\mathbf{e}_q+A_\perp'(\mathbf{q})\mathbf{e}_\perp'+A_t(\mathbf{q})\mathbf{e}_t \tag{2.14}$$

and

$$\mathbf{B}(\mathbf{q})=B_z(\mathbf{q})\mathbf{e}_z+B_\perp(\mathbf{q})\mathbf{e}_\perp+B_t(\mathbf{q})\mathbf{e}_t \ , \tag{2.15}$$

where the orthonormal set for $\mathbf{A}(\mathbf{q})$ is

$$\mathbf{e}_q=\mathbf{q}/q \ , \tag{2.16a}$$

$$\mathbf{e}_t=\frac{\mathbf{m}_0\times\mathbf{q}}{|\mathbf{m}_0\times\mathbf{q}|} \ , \tag{2.16b}$$

$$\mathbf{e}_\perp'=\mathbf{e}_q\times\mathbf{e}_t \ , \tag{2.16c}$$

while the orthonormal set for $\mathbf{B}(\mathbf{q})$ is

$$\mathbf{e}_z\equiv\mathbf{m}_0 \ , \tag{2.17a}$$

$$\mathbf{e}_\perp=\mathbf{q}_\perp/q_\perp \ , \tag{2.17b}$$

$$\mathbf{e}_t=\mathbf{e}_z\times\mathbf{e}_\perp \ . \tag{2.17c}$$

Note that $A_q(\mathbf{q})$ is the component of $\mathbf{A}(\mathbf{q})$ along $\mathbf{q}$, i.e., a splay mode, while $A_\perp'(\mathbf{q})$ is the component of $\mathbf{A}(\mathbf{q})$, orthogonal to $\mathbf{q}$, in the $\mathbf{q}$-$\mathbf{m}_0$ plane, and then it is a bend mode; finally $A_t(\mathbf{q})$ is the component of $\mathbf{A}(\mathbf{q})$ perpendicular to the $\mathbf{q}$-$\mathbf{m}_0$ plane, and then it is a mixing of bend and twist modes (see also Fig. 1).

In the discotic gauge, $\delta\mathbf{m}(\mathbf{q})$ is given by

$$\delta\mathbf{m}(\mathbf{q})=\delta m_t(\mathbf{q})\mathbf{e}_t+\delta m_\perp(\mathbf{q})\mathbf{e}_\perp \ , \tag{2.18}$$

since $\delta m_z=0$, as $\delta\mathbf{m}$ is orthogonal to $\mathbf{m}_0$. In order to transform from the discotic gauge to the soft-bend gauge, in Fourier space, we have to take

$$A_\perp'(\mathbf{q})=0 \ . \tag{2.19}$$

However, even if the pure-bend component of $\mathbf{A}(\mathbf{q})$ vanishes, Eq. (2.19), the critical field $\mathbf{A}(\mathbf{q})$ still contains bend, since its component $A_t(\mathbf{q})$ is a mixing of bend and twist modes. In fact, taking Eq. (2.19) into account, the Fourier transform of the bend term $(\mathbf{m}_0\times\operatorname{rot}\mathbf{A})^2$ is $q_z^2|A_t(\mathbf{q})|^2$, while the Fourier transform of the twist term $(\mathbf{m}_0\cdot\operatorname{rot}\mathbf{A})^2$ is $q_\perp^2|A_t(\mathbf{q})|^2$. Therefore, if we took bend vanishing, that would lead to $A_t(\mathbf{q})=0$ and then even twist would vanish. The director field, in the soft-bend gauge, is thus

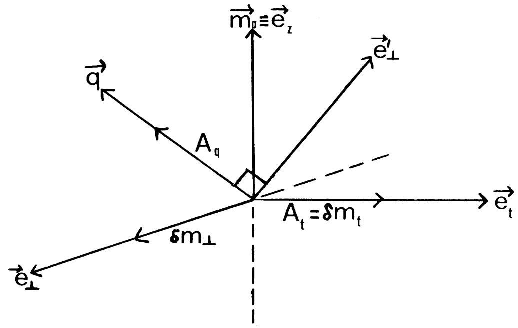

FIG. 1. Unperturbed director $\mathbf{m}_0$ is along the $\hat{\mathbf{z}}$ axis. In the discotic gauge, variations $\delta\mathbf{m}$ in $\mathbf{m}$ must occur in the plane orthogonal to $\mathbf{m}_0$. In this connection, the orthonormal triad $\{\mathbf{e}_z,\mathbf{e}_\perp,\mathbf{e}_t\}$, defined in Eq. (2.17) of the text, is displayed. In the soft-bend gauge, on the other hand, the two independent components of $\mathbf{A}$ are $A_t=\delta m_t$, which is orthogonal to the $\mathbf{m}_0-\mathbf{q}$ plane, and A_q, which lies along $\mathbf{q}$. The third component of $\mathbf{A}$, lying in the $\mathbf{m}_0$-$\mathbf{q}$ plane perpendicular to $\mathbf{q}$, vanishes. In this connection, the orthonormal triad $\{\mathbf{e}_q,\mathbf{e}_t,\mathbf{e}'_\perp\}$, defined in Eq. (2.16) of the text, is displayed.

$$\mathbf{A}(\mathbf{q})=A_q(\mathbf{q})\mathbf{e}_q+A_t(\mathbf{q})\mathbf{e}_t \ . \tag{2.20}$$

The difference between the two gauges is displayed in Fig. 1. In smectics, an analogous gauge transformation, anyway different, in form, leads to vanishing splay in the critical field ($A_q=0$), so that $\mathbf{A}$ is a pure transverse field [9].

In addition to Eq. (2.19), the Fourier transforms of Eqs. (2.8) and (2.9) fix the soft-bend gauge, in Fourier space:

$$q_zB_z(\mathbf{q})+q_\perp B_\perp(\mathbf{q})=0 \tag{2.21}$$

and

$$q_\perp B_t(\mathbf{q})=0 \ , \tag{2.22}$$

respectively. Moreover, using $\delta m_z=0$ and Eq. (2.20), we get

$$B_z(\mathbf{q})=-A_z(\mathbf{q})=-\frac{q_z}{q}A_q(\mathbf{q}) \ . \tag{2.23}$$

By Eqs. (2.21)–(2.23), we obtain

$$B_t(\mathbf{q})=0 \ , \tag{2.24a}$$

$$B_\perp(\mathbf{q})=\frac{q_z^2}{q_\perp q}A_q(\mathbf{q}) \ , \tag{2.24b}$$

so that the field $\mathbf{B}(\mathbf{q})$, Eq. (2.15), in the soft-bend gauge, is given in terms of $\mathbf{A}(\mathbf{q})$:

$$\mathbf{B}(\mathbf{q})=\left[-\frac{q_z}{q}\mathbf{e}_z+\frac{q_z^2}{q_\perp q}\mathbf{e}_\perp\right]A_q(\mathbf{q}) \ . \tag{2.25}$$

The Fourier transform of the bend term $(\mathrm{rot}\mathbf{B})^2$ is then $(q^2q_z^2/q_\perp^2)|A_q(\mathbf{q})|^2$, which must be added to $q_z^2|A_t(\mathbf{q})|^2$, coming from the Fourier transform of $(\mathbf{m}_0\times\mathrm{rot}\,\mathbf{A})^2$. The splay term $(\mathrm{div}\,\mathbf{A})^2$ gives, in Fourier space, $q^2|A_q(\mathbf{q})|^2$, while the twist term $(\mathbf{m}_0\cdot\mathrm{rot}\,\mathbf{A})^2$ gives $q_\perp^2|A_t(\mathbf{q})|^2$. Therefore, the Frank energy (2.2c), in Fourier space, can be reexpressed in terms of the soft-bend gauge as

$$F_{\mathrm{el}}=\tfrac{1}{2}\int\frac{d^3q}{(2\pi)^3}\left\{(K_2q_\perp^2+K_3q_z^2)|A_t(\mathbf{q})|^2+\left[K_1q^2+K_3\frac{q^2q_z^2}{q_\perp^2}\right]|A_q(\mathbf{q})|^2+(\gamma_1q_z^2+\gamma_2q_\perp^2)|\Omega_z(\mathbf{q})|^2+\gamma_3q_zq_\perp[A_t(\mathbf{q})\Omega_z^*(\mathbf{q})+\mathrm{c.c.}]\right\} \ . \tag{2.26}$$

Finally, we notice that, as a consequence of the performed bend-decoupling gauge transformation, a term nonanalytic in q, whose coefficient is just the bend constant K_3, appears in Frank energy (2.26). In Sec. III we shall see that such a nonanalytic property implies noncritical behavior of K_3.

III. THE GAUGE-FIELD PROPAGATORS

The application of the equipartition theorem to (2.26) gives the gauge-field propagators in terms of the soft-bend gauge. Taking free energy in terms of the K_BT unit, the free, i.e., not coupled to the matter, propagators of the gauge fields are

$$\langle|A_t(\mathbf{q})|^2\rangle=\frac{1}{K_2q_\perp^2+K_3q_z^2} \ , \tag{3.1a}$$

$$\langle|A_q(\mathbf{q})|^2\rangle=\frac{1}{K_1q^2+K_3q^2\frac{q_z^2}{q_\perp^2}} \ , \tag{3.1b}$$

$$\langle|\Omega_z(\mathbf{q})|^2\rangle=\frac{1}{\gamma_1q_z^2+\gamma_2q_\perp^2} \ , \tag{3.1c}$$

where we have taken, for simplicity, $\gamma_3=0$. However, the propagator (3.1b), which is our main task, is exact even for $\gamma_3\neq0$, as γ_3 couples A_t with Ω_z, while the component A_q remains decoupled, as one can see from (2.26). The bend term in $\langle|A_q(\mathbf{q})|^2\rangle$, Eq. (3.1b), is not analytic in q, which is due to the previously performed gauge transformation that has decoupled bend from the matter.

The propagator (3.1a) of A_t is the same as that of δm_t, which can be drawn from (2.12), being $A_t=\delta m_t$. On the other hand, A_q can be related to $\delta m_\perp$. In fact, using

$$\delta m_\perp(\mathbf{q})=\mathbf{e}_\perp\cdot\mathbf{A}(\mathbf{q})+B_\perp(\mathbf{q}) \ , \tag{3.2}$$

and exploiting also (2.20) and (2.24b), one obtains

$$\delta m_\perp(\mathbf{q})=\frac{q}{q_\perp}A_q(\mathbf{q}) \ , \tag{3.3}$$

and then

$$\langle |\delta m_\perp(\mathbf{q})|^2\rangle = \left[\frac{q}{q_\perp}\right]^2 \langle |A_q(\mathbf{q})|^2\rangle = \frac{1}{K_1 q_\perp^2 + K_3 q_z^2}, \tag{3.4}$$

which can be also drawn directly from (2.12). Note that, in Eq. (3.4), the first equality is always valid, while the second equality holds good only for free propagators.

Defining, in the discotic gauge,

$$D_{\alpha\beta}(\mathbf{q}) = \langle \delta m_\alpha(\mathbf{q}) \delta m_\beta^*(\mathbf{q})\rangle, \tag{3.5}$$

with $\alpha,\beta = x,y$, the inverse, in the matrix sense, of the free propagator tensor is (for $\gamma_3 = 0$)

$$Q^0_{\alpha\beta}(\mathbf{q}) = (K_1 q_\perp^2 + K_3 q_z^2) e_{\perp\alpha} e_{\perp\beta} + (K_2 q_\perp^2 + K_3 q_z^2) e_{t\alpha} e_{t\beta}. \tag{3.6}$$

The coupling between the order parameter and the gauge field $\delta\mathbf{m}$ renormalizes the inverse propagator. At one-loop order in the perturbative series of graphs, the contribution due to the fluctuations of the order parameter is [1,11]

$$\tilde{Q}_{\alpha\beta}(\mathbf{q}) = \frac{q_0^2}{16\pi} \xi_\parallel q_\perp^2 \delta_{\alpha\beta}, \tag{3.7}$$

where $\xi_\parallel = (2aM_\parallel)^{-1/2}$ is the correlation length of the order parameter in the direction parallel to the direction of the liquid columns, while $\delta_{\alpha\beta} = e_{\perp\alpha} e_{\perp\beta} + e_{t\alpha} e_{t\beta}$ is the Kronecker delta in the XY plane. The inverse of $D_{\alpha\beta}$, Eq. (3.5), at one-loop order is then

$$\begin{aligned} Q_{\alpha\beta}(\mathbf{q}) &= Q'_{\alpha\beta}(\mathbf{q}) + \tilde{Q}_{\alpha\beta}(\mathbf{q}) \\ &= \left[\left[K_1 + \frac{q_0^2}{16\pi}\xi_\parallel\right] q_\perp^2 + K_3 q_z^2\right] e_{\perp\alpha} e_{\perp\beta} \\ &\quad + \left[\left[K_2 + \frac{q_0^2}{16\pi}\xi_\parallel\right] q_\perp^2 + K_3 q_z^2\right] e_{t\alpha} e_{t\beta}. \end{aligned} \tag{3.8}$$

In the soft-bend gauge, the inverse, in the matrix sense, of the free propagator tensor $\langle A_\alpha(\mathbf{q}) A_\beta^*(\mathbf{q})\rangle$, is

$$P^0_{\alpha\beta}(\mathbf{q}) = \left[K_1 q^2 + K_3 q^2 \frac{q_z^2}{q_\perp^2}\right] e_{q\alpha} e_{q\beta} + [K_2 q_\perp^2 + K_3 q_z^2] e_{t\alpha} e_{t\beta}, \tag{3.9}$$

while the corresponding renormalized inverse propagator, in accordance with Eq. (3.4), is, at one-loop order,

$$\begin{aligned} P_{\alpha\beta}(\mathbf{q}) &= \left[\left[K_1 + \frac{q_0^2}{16\pi}\xi_\parallel\right] q^2 + K_3 \frac{q^2 q_z^2}{q_\perp^2}\right] e_{q\alpha} e_{q\beta} \\ &\quad + \left[\left[K_2 + \frac{q_0^2}{16\pi}\xi_\parallel\right] q_\perp^2 + K_3 q_z^2\right] e_{t\alpha} e_{t\beta}. \end{aligned} \tag{3.10}$$

Near the second-order phase-transition temperature, the correlation length $\xi_\parallel$ diverges, so driving the critical enhancements of the elastic constants. Actually, the transition from the hexatic $N+6$ phase to the hexagonal discotic phase can be weakly first order [1], so that pretransitional effects can be observed. From Eq. (3.8) or (3.10), we obtain, for the renormalized elastic constants, at one loop order [1,11],

$$K_1' = K_1 + \frac{q_0^2}{16\pi}\xi_\parallel, \tag{3.11a}$$

$$K_2' = K_2 + \frac{q_0^2}{16\pi}\xi_\parallel, \tag{3.11b}$$

$$K_3' = K_3. \tag{3.11c}$$

Indeed, the equality (3.11c) for K_3 is exact to all orders in perturbation theory. In fact, there are no graphs that renormalize K_3, since it is the coefficient of a term nonanalytic in q, in the propagator (3.9) that is drawn from free energy (2.26). The perturbation expansion only yields analytic terms and therefore it cannot reproduce a nonanalytic term. For a similar argument, in smectic phases the splay constant K_1 is noncritical [9,10]. We stress that such a nonperturbative result originates in the previously performed gauge transformation, which has decoupled bend from the matter.

In conclusion, we have studied a peculiar gauge theory, Abelian but not equivalent to electrodynamics, for the conjectured hexatic $N+6$ phase near the transition to the hexagonal discotic phase, on the basis of our model for that phase transition [1,2]. The full curvature field $\mathbf{\Omega}$, which fixes the local orientational order [1], plays the role of a gauge field. The main result is that it is possible to define a suitable gauge transformation, which decouples the bend mode of the director distortion from the density-wave order parameter. The bend component of the gauge field, being decoupled from the order parameter, is not affected by the renormalization yielded by the gauge coupling. As a consequence, the bend elastic constant K_3 is noncritical. On the other hand, the proposed gauge transformation makes the bend term in the free energy nonanalytic in the wave vector, so that the elastic constant K_3 is not renormalized to all orders in perturbation theory. Indeed, such a theory is not strictly gauge invariant (see also [4]), as the bend term breaks the local invariance under rotations. Anyway, just for this reason, the director gauge field $\delta\mathbf{m}$ can be separated, by a gauge transformation, into the superposition of a critical mode and a noncritical bend mode. Such a peculiarity of the bend mode is expressed in the free energy, where, in terms of what we have called the soft-bend gauge, the energy of bend is not analytic in the wave vector. Therefore, the noncritical behavior of bend goes beyond the first-order approximation in perturbation theory, in terms of which the results (3.11) were drawn in [1], and it must be considered as an exact, nonperturbative result that can be argued on the basis of the above-described gauge theory for discotic phases.

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